Middle School Mathematics Comparisons for Singapore Mathematics, Connected Mathematics Program, and Mathematics in Context: A Summary

(Including Comparisons with the NCTM Principles and Standards 2000)

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1 Introduction

In the mid-nineties, the mathematical community in the United States was severely shaken by the results of the Third International Math and Science Study (TIMSS). The test performance of our students dropped from mediocre at the elementary level through lackluster at the middle school level and down to truly distressing at the high school level. Considerable scrutiny followed, and is still following. Several clear mandates were generated by the study. One was to look closely at mathematics in Singapore, since of the forty-one nations' students tested, Singapore's scored at the very top. Exploration revealed a series of textbooks with a particularly nice development of the mathematical structure. This led to the obvious question: How do Singapore's texts compare with the books which have been designed since the early nineties in the United States, books that represent the direction of the current thrust in American mathematical education? This study is an effort to supply some data and general information toward such a comparison. We were asked to compare the Singapore mathematics materials at the middle school level with two American curricula. We chose two highly regarded and widely used curricula that were both designed to meet the 1989 NCTM Curriculum Standards (http://www.nctm.org/standards): the Connected Mathematics Program (CMP) and Mathematics in Context (MIC).

Before launching into a description of what we have done, it seems wise to mention some of the limitations imposed by the situation, lest we appear to be claiming to have achieved things which the circumstances preclude. To begin with, we are in some sense comparing the incomparable. The Singapore series is intended to be used in the way that textbooks have traditionally been used, under the authority of a traditional teacher. The two curricula we compare it with are based on a quite different philosophy of how learning occurs, and as a result are differently designed. This led to some tough decisions in designing the study, as well as to some places within the study where a caveat is necessary.

We have decided in this study to perform two different kinds of comparisons. First we compare each curriculum with the National Council of Teachers of Mathematics (NCTM) Principles and Standards 2000. Secondly we compare the three curricula with each other, not confined by the NCTM Principles and Standards. The first is admittedly not a fair comparison for the Singapore curriculum, as the two American curricula were designed from the ground up to meet the 1989 NCTM Curriculum Standards, while Singapore's curriculum was designed with Singapore's GCE (General Certificate of Education) examinations in mind. Nevertheless, if a school district in the United States would like to evaluate the Singapore curriculum (or any curriculum) for adoption, then it is important that we provide the information on how it fares against a set of adopted American standards. For example, any curriculum which omits a particular mathematical strand, or which assumes that the teachers are mathematics specialists, or which stereotypes females is probably not viewed by most as appropriate for adoption in American schools. In the second kind of comparison, we draw upon our expertise as mathematicians to evaluate the accuracy, depth, and breadth of the mathematics of one curriculum against another. For example, we will point out cases where two curricula satisfy an NCTM standard although the level of mathematics in one of them is one or two grade levels lower than the other. We also comment on the level and quality of teacher and parent support the curriculum provides, and discuss the difficulties of implementation.

Other studies of American middle school curricula have been done using a different set of criteria. Project 2061 (http://www.project2061.org/matheval/index.htm) is a notable example. The findings of that project placed CMP and MIC among the top of the thirteen curricula studied. Other critiques of curricula from university mathematicians and parents can be found at the Mathematically Correct website at (http://www.mathematicallycorrect.com). We feel, however, that the 2000 NCTM Principles and Standards are widely known by the various communities that will read our report, are readily available (see http://www.nctm.org), and present a more comprehensive structure and balanced perspective. For example, Project 2061 uses six benchmark questions to evaluate twelve curricula as to content and a large number of pedagogical issues. We use 72 questions to compare the curricula against the 10 overarching standards, distributed as: number (14), algebra (9), geometry (12), measurement (9), data and probability (10), problem solving (4), reasoning and proof (4), communication (4), connection (3), and representation (3). In addition, we use 13 questions to compare the curricula with the 6 principles, distributed as: equity (4), curriculum (3), teaching (1), learning (2), assessment (2), and technology (1).

Furthermore, the 2000 NCTM Principles and Standards represent a tremendous amount of thoughtful effort from a community with a wide variety of expertises. The choices they have made are ones to which we can comfortably subscribe, whether or not we agree with them in every detail. To reinvent that particular wheel was not the goal of this study. The recent article in the October 2000 issue of the *Notices of the American Mathematical Society* further supports the choice of the NCTM Principles and Standards 2000 for our study.

Another point to consider is our expertise, both what it is and what it is not. The group that created this comparison consists entirely of people who combine high-level training in mathematics with an interest in education, but who have neither direct experience of teaching in the American K-12 classroom nor the training for it. We have tried, therefore, to distinguish consistently between the statements that we make with genuine authority based on our grounding in mathematics and those statements that we sometimes permit ourselves based on a certain amount of related experience and a lot of observation and study of how teaching and learning can best occur in K-8 classrooms.

The specific structure by which we have proceeded is the following: one of the three faculty members in our group spent many weeks going through every chapter or module of the three curricula being compared, deciding how they met each of the criteria encapsulated in the questions we extracted from the Principles and Standards, and assigning to each question a corresponding score with an explanatory note. When she had finished, each of the three graduate students in the group chose one of the curricula and went over it in the same kind of detail, finding in particular the areas where they questioned the score or the discussion of it. Meanwhile the other two faculty members did a less detailed but more overarching study of the entire set of scores and discussions. There ensued a series of meetings by pairs and trios to clarify or negotiate differences of opinion with regard to the scoring and discussion. In this portion of the study we felt most strongly that we were relying on the knowledge we share in the field of mathematics.

At the same time, each of the graduate students wrote a summary of his impressions of the curriculum in which he had specialized. Each of these students has spent months working in various capacities (generally within classrooms) with elementary teachers who were beginning to teach from the 5th and 6th grade Everyday Mathematics curriculum. In addition, the graduate student who specialized in the Singapore curriculum traveled to Singapore for the purpose of observing classes and talking with teachers and educational administrators. To finish, we wrote a summary of our joint impressions and conclusions that includes comparisons of the curricula with the NCTM standards and directly with each other.

Before we describe the structure of the report itself, we need to make very clear and explicit one of our working hypotheses. The question of what a student learns depends heavily on the teacher. Different styles of teaching bring out different kinds of learning and produce different possibilities for non-learning. On none of these implementation issues can we possibly base our report. We have therefore adopted the universal assumption for each of the curricula that the lessons go exactly as planned, and that the students learn all of what each lesson is set up to offer.

The rest of the report is as follows. Chapter 2 contains brief individual summaries of backgrounds and impressions of each of the three curricula. Chapters 3 and 4 contain the comparisons with the 2000 NCTM Standards and the 2000 NCTM Principles, respectively. The "bullets" that describe the mathematical requirements of each standard (see, for example, the head of the Number Standard at http://standards.nctm.org/document/chapter6/numb.htm) are used as our questions. We assign a score between 0 and 3 (0 being the lowest) to each question and then give reasons for the score. We include a short summary for each curriculum after each of the ten overarching standards. The same type of analysis is done for each of the six principles. We include a short summary for each curriculum at the end of Chapter 4. Most of the questions for Chapter 4 were taken from the *Guidebook to Examine School Curricula – TIMMS as a Starting Point to Examine Curricula, Attaining Excellence Report*, U.S. Department of Education, Office of Educational Research and Improvement (available at http://timss.enc.org/topics/timss/kit). Other questions were made by the project team to cover issues in the 2000 NCTM Principles that were not addressed in the *Guidebook* above. Chapter 5 contains direct comparisons of the curricula using the information gained from Chapters 3 and 4 in conjunction with our joint expertise as mathematicians and applied mathematicians. Chapter 6 contains our conclusions.

This report is a shortened version of the November 2, 2000 report with the same title that has been submitted to NSF. The full report gives the detailed evidence for each of the questions for each curriculum. Readers that want to see explicitly how the mathematics builds up through the grade levels in each curriculum and the basis of our scoring of each curriculum are encouraged to read the full report. The full report also contains Appendices A, B, and C which are stand-alone reports on the Singapore curriculum, CMP, and MIC, respectively. These appendices consist of the corresponding material from Chapters 2, 3, and 4, sorted by curriculum. The full report and the three Appendices are posted on the WEB at http:/www.amath.washington.edu/~adams/comparisons.html.

2 Background and General Impressions

2.1 The Singapore Curriculum

Before we can make objective comparisons between American and Singapore mathematics curricula, we must first point out some differences between the educational systems in both countries. Although the purpose of this report is not to critique either educational system or to offer explanations for the TIMSS results, neglecting to pursue this larger perspective gives the impression that the math curriculum bears the entire responsibility for students' education. Instead, we recognize that the mathematics curriculum is only one component of a child's education, which includes among other components teachers, parents, peers, government, and culture.

One member of our team had the opportunity to travel to Singapore in February 2000 to find out more about their education system. During this visit, Singapore's Ministry of Education (MOE) graciously allowed him to talk to its Curriculum Planning and Development Division (CPDD), and to visit several secondary schools. Here is what he learned:

In Singapore, students spend 6 years in primary school (corresponding to American grades 1–6), followed by 4 years in secondary school (corresponding to American grades 7–10). At the end of Primary 6, students' abilities are assessed using the Primary School Leaving Examination (PSLE) and students assigned to one of three streams: Special, Express, and Normal (broken further into two sub-streams, Normal-Academic and Normal-Technical).

The differences between the three streams are largely in their educational goals. The Special and Express streams prepare students to take the General Certificate of Education (GCE) "O" Level exam at the end of their fourth year of secondary school, [1] whereas the Normal stream prepares students for the GCE "N" Level exam. (If stu-

[1] This pace represents a one-year acceleration of the traditional British system, on which it is based. In the British system, the CGE "O" Level exam is taken at the end of the fifth year of secondary school.

dents in the Normal stream do well on the "N" exam, they are allowed to study an extra year to prepare for the "O" level exam.) Students who pass the "O" Level exam are allowed to pursue "pre-university" education (corresponding to American grades 11–12); students who pass the "N" Level exam can apply to technical and vocational schools. Which exam a student passes (and prepares for), therefore, is extremely important because it determines the careers the student can pursue. In effect, by the end of the sixth grade, Singapore's education system streams students by their academic ability, predisposing them towards certain vocations. Not surprisingly, we observed that, in general, Singapore's students are more motivated to excel academically than their American counterparts. It is not uncommon for Singapore students to stay after school for enrichment lessons, attend supplemental programs, or hire tutors at a much higher extent than in the United States.

Naturally, the three streams also differ in their difficulty and amount of coursework. Typically, students in the Special, Express and Normal-Academic streams spend 2.5 to 3 hours a week on math in the classroom. According to the CPDD, students in the Normal-Technical stream spend 4 to 5 hours a week, as they need more attention.

Only three subjects are included on the Primary School Leaving Examination (PSLE): language, math and science. Teachers at the primary level typically teach all three subjects and serve as the "form teacher" (or "homeroom teacher"), who is responsible for the overall care of a child. Secondary teachers usually teach two subjects (commonly math and computer science or math and language). Singapore's educational system at the secondary level, then, is largely administered by teachers who are content-area specialists.

The most striking difference between the educational systems of both countries, however, is in governmental support of education. For example, the government encourages every teacher to attend at least 100 hours of training each year, and has developed an intranet called the "Teachers' Network," which enables teachers to share ideas with one another. Largely due to the fact that Singapore has a small geographic area, its government has a much greater ability to provide uniform educational experiences for its students than the United States government.[2] Given that all teachers receive their training from the National Institute of Education, the uniformity in teaching styles that we observed in the four classrooms during our February visit becomes less surprising.

It is this uniformity of their education system that allows us even to speak of a "Singapore mathematics curriculum," because there is only one mathematics curriculum (developed by the CPDD) used by all public primary schools. Nevertheless, we must carefully define what we mean by "Singapore mathematics curriculum" in this report. Because the middle school grades (6–8) in the United States correspond to the last year of primary school (Primary 6) and the first two years of secondary school (Secondary 1–

[2]Mr. Soh Cheow Kian, assistant director of the CPDD's sciences group, told us that he believes the main differences between Singapore and the United States are teaching approaches and government support.

2), we must consider both primary and secondary Singapore math curricula. While there is only one mathematics curriculum for Singapore primary schools, there are a handful of choices at the secondary level. The CPDD only sets guidelines for secondary math curricula—independent publishers are responsible for developing the teaching materials. The mathematics department of each secondary school is free to select from the available curricula. Furthermore, there are different mathematics curricula for the different streams. So to simplify matters, we decided to look at the curricula specifically designed for the Express stream, since most students (about 60%) enter this stream. This decision narrowed our focus to two math curricula, the *New Elementary Mathematics* series published by Pan Pacific, and *Syllabus D Mathematics* series published by Shing Lee. From the interviews with our CPDD hosts and the secondary teachers we observed during the February visit, we noticed that the problem solving strategies and problems in the Shing Lee texts are more popular and favorably viewed.

To summarize, when we refer in this report to Singapore's mathematics curriculum at the grade levels corresponding to the American middle school grades, we refer specifically to the Primary 4–6[3] materials (*(4A)* and *(4B)*; *(5A)* and *(5B)*; *(6A)* and *(6B)*), and the Secondary 1–2 materials published by Shing Lee (*(SL1)* and *(SL2)*). In our discussion, we will also occasionally refer to the Secondary 1–2 materials by Pan Pacific. The Secondary 3–4 materials by Shing Lee (*(SL3)* and *(SL4)*) corresponding to ninth and tenth grades, respectively, are also referred to occasionally to supply the reader with further information, but are not part of our study. We examined the student texts, the student workbooks and the Teacher's Guides for the *(4A)* - *(6B)* levels and the student texts, the student workbooks, and the Teacher's Resource Manual for the *(SL1)* and *(SL2)* levels.

The most important fact to remember about Singapore's math curriculum is that it was designed to be used in Singapore, not the United States. Singapore's math curriculum for primary schools is designed to prepare students for the PSLE, and likewise its secondary school curriculum for the GCE Examinations. Obviously, these curricula are not likely to fare well if we compare them against American standards, because they weren't designed to meet them. Nevertheless, there is some opinion[4] that America should start using Singapore's math curriculum. As a first step towards serious consideration of this viewpoint, we must determine how well Singapore's curriculum meets the standards for mathematics education developed by the some of the best teachers in our country.

In addition, we believe that there is a fundamental difference in what a math curriculum is expected to do in each country. We believe that the designers of Singapore's secondary math curricula expect their materials to serve as reference books, with lots of practice questions, for their students. Through our observations and interviews, we

[3]Students typically spend the last 30% of Primary 6 preparing for the PSLE, so the amount of math content contained in the Primary 6 mathematics book is lower than in the books from previous levels. To compensate, we carefully looked through the Primary 4 and 5 textbooks as well.

[4]One example is Cheryl Corley's report on NPR's *Morning Edition*, June 8, 2000.

noticed that Singapore secondary teachers share this conception of their math curricula's function because they have become adept at collaborating with their peers and drawing on multiple sources of materials. Singapore's secondary math curricula do not provide many instructional hints or suggestions for assessment, but that is because there is little expectation for those types of materials. (Singapore's primary math curriculum, however, comes with excellent teachers' guides, which give the teacher instructional help, outline the ways the mathematical ideas connect throughout the curriculum, and provide articulation across the primary grades. This could be because the primary teachers are not mathematics specialists, and there is a perceived need at this level.)

In contrast, our great diversity of teacher support structures and teacher expertise levels in the U. S. prompts curriculum designers to create curricula that try to be many things to many people. For better or worse, American teachers have come to expect well-designed, thorough instructional materials from curriculum makers. Therefore, we believe that many American school districts will find the lack of substantial printed guidance for teachers to be a significant deterrent against the adoption of Singapore's secondary math curricula.

One of the most attractive features of Singapore's math curricula is that the student books contain many problems that are worked out explicitly. Often, the examples that are worked out demonstrate the necessary steps in great detail, although rarely does one find alternative methods of solving problems. In addition, the Singapore math curricula have many practice problems for students. Singapore's primary math curriculum and the Shing Lee secondary curriculum's workbooks contain additional practice for the student.

The presentation in the Primary 4–6 books includes concrete examples, pictorial representations, cartoons of children explaining how to think about the topic, and finally the general case. The problems are short word problems that assume a grade-appropriate reading level. The presentation in the Secondary *(SL1)* and *(SL2)* books follow the form of a standard mathematics text. The topic is introduced, terms are defined, examples are given to illustrate the recommended strategies, and word problems are given for students to practice. The word problems and presentation assume a grade-appropriate reading level. The textbooks are careful to indicate new mathematical terminology by displaying it in bold or italics in the text, but they do not encourage students to actively use this terminology in the problems they do.

The materials that we studied do not involve technology in a significant way. Besides the use of calculators, the Department of Educational Technology (CDIS) computer software accompanies the Primary 4–6 materials, but we did not have access to this software. The student materials for Primary 6 do not include references to CDIS, but the teacher's manual frequently gives cross-references to it.

The quality and accuracy of the mathematics in Singapore's textbooks is high. There are a few notable examples where the mathematics in Singapore's textbooks is at a level of difficulty that exceeds NCTM expectations for the middle grades. In Primary 6, students solve complicated word problems involving proportions, without using algebra. In Secondary 1 and 2, students get a lot of practice manipulating algebraic expressions, and perform some difficult triangle congruence proofs. However, Singapore's curriculum does not introduce students to statistics until Secondary 2, and does not include any probability in the equivalent of the American primary or middle school grades.

While the mathematics in Singapore's curriculum may be considered rigorous, we noticed that it does not often engage students in higher-order thinking skills. When we examine the types of tasks that the Singapore curriculum asks students to do, we see that Singapore's students are rarely, if ever, asked to analyze, reflect, critique, develop, synthesize, or explain. The vast majority of student tasks in the Singapore curriculum is based on computation, which primarily reinforces only the recall of facts and procedures. This bias towards certain modes of thinking may be appropriate for an environment in which students' careers depend on the results of a standardized test, but we feel it discourages students from becoming independent learners.

We also point out some unbalanced gender references in Singapore textbooks. In particular, the textbooks published by Shing Lee and Pan Pacific provide unnecessarily stereotypical depictions of men and women in their word problems, cartoons, and textbook prose.

Furthermore, we believe that Singapore's curriculum does not adequately recognize that students have a wide range of learning styles. For example, Singapore's curriculum does not recognize that some students learn better through guided discovery than a direct presentation of concepts and procedures. Singapore's curriculum seems to be based on the view that the teacher is the primary disseminator of information in the classroom.

To summarize, our overall opinion of Singapore's mathematics curriculum is that its educational objectives are not well aligned with those in the NCTM standards. Singapore's mathematics curriculum does an excellent job of helping students acquire mathematical facts and procedures, through the many worked-out examples and large numbers of practice problems in its textbooks. However, the NCTM has identified other goals of mathematics education that are largely missing in Singapore's curriculum, most notably reasoning, communication, and connections (a larger conception of mathematics as a coherent whole that interacts with our world). Furthermore, Singapore's secondary math curricula provide very little support for teachers.

2.2 The Connected Mathematics Program

The Connected Mathematics Project (CMP) was developed by Glenda Lappan and others at Michigan State University and funded by the National Science Foundation. The 1998 edition was published by Dale Seymour Publishers. The curriculum was developed to be in line with the pedagogy and content in the 1989 National Council of Teachers of Mathematics (NCTM) standards, namely the *Curriculum and Evaluation Standards* *for School Mathematics*, the *Professional Standards for Teaching Mathematics*, and the *Assessment Standards for School Mathematics.* More information on CMP can be found at their website (http://www.math.msu.edu/cmp/index.html).

CMP focuses on mathematical content in the number, geometry, measurement, algebra, statistics and probability strands. The "Getting to Know CMP" guide, which comes with each grade's books, stresses that CMP students use the processes of counting, visualizing, comparing, estimating, measuring, modeling, reasoning, playing and using tools. Each of the grade levels has eight modular units. Some of these units have titles such as *How Likely Is It?* (probability) and *Growing, Growing, Growing* (exponential growth) which allow the teacher, students, and parents to get a sense of the mathematical content. A Teacher's Guide and student book is provided for each of the units.

The Teacher's Guides contain all the pages of the student book numbered in a manner consistent with the teacher pages, a "Teaching the Investigation" section, different types of assessments, additional problems, samples of student work, articulation information for the teacher, blackline masters, and form letters to parents in both English and Spanish describing the purpose of each unit and how they can best support their child's mathematical development at home. The "Teaching the Investigation" sections are the heart of the CMP curriculum. They give the teacher guidance on how to teach the lesson, an explanation of the mathematics in the lesson, and specific questions to ask students to make sure the important mathematical points are brought out during class. In the absence of a mentor teacher in each building, the assistance provided by the "Teaching the Investigation" sections could be very valuable to teachers that are not yet comfortable with the mathematics or with the discovery method of teaching. Even though CMP provides enough guidance to support a novice teacher, an experienced teacher can use his or her own creativity to supplement lessons and to meet the individual needs of students.

A CMP lesson, called an *Investigation*, is organized in three parts: *Launch*, *Explore*, and *Summarize*. *Launch* is the lesson introduction; it includes definitions, explanation of relevant concepts and other background material. *Explore* encourages students to work individually, then in pairs or groups, while the teacher circulates through the classroom and listens to students. *Summarize* allows for groups to share their findings followed by a class discussion. The typical lesson concludes with a "Mathematical Reflections" section, enabling students to reflect on their own learning. CMP emphasizes a discovery-based approach to learning that encourages students to select, adapt, and analyze problem solving strategies in order to develop mathematical understanding and become autonomous learners.

The curriculum provides numerous projects and problems for students. "ACE" problems appear at the end of each Investigation. These are grouped into Applications, Connections and Extensions. The Application problems reinforce the ideas currently being studied. The Connection problems integrate these ideas with strands that have been previously taught. The Extension problems are often the most challenging and carry the ideas forward. The Teacher's Guides also contain a *Question Bank* with additional problems that can be assigned to students that require more practice. Most of the twenty-four books in the curriculum have a Unit Project that requires the students to use the mathematics they have learned in pursuit of a larger goal. Students also have journals in which they can write their thoughts and record their work. The curriculum suggests that teachers should check these journals for completeness and not for correctness, so students are free to express their thinking. By examining these records, both correct and incorrect, the teacher is better able to assist students and assess the progress of his or her class.

The curriculum is constructed around five instructional themes: teaching for understanding ("big ideas"), connections, investigations, representations and technology. For example, CMP presents mathematics in a coherent way with an emphasis on connections among the mathematical ideas - thus the title of the curriculum. Students are urged to use multiple representations. For example, in the section *Solving Linear Equations* from the 8^{th} grade book *Say It With Symbols*, students are asked to compare graphical, tabular, and symbolic representations of a linear function. Regarding the technology theme, CMP 6^{th} grade students are required to have a standard "four-function" calculator and 7^{th} and 8^{th} grade students utilize graphing calculators with statistical capabilities. Some computer programs are included with the curriculum for enhancing probability and geometry lessons. We find that in most cases these technologies did not replace pencil and paper arithmetic, but since students themselves choose when to use calculators, a dependence on the calculator for problems of too low a level could develop.

Turning to the mathematical content of CMP, we find that most of the concepts presented in the number strand are a review for students that have gone through, for example, the Everyday Mathematics curriculum for grades K-6. The number strand is arguably the most basic and fundamental mathematics strand and much of the presentation in CMP is below the level articulated in the 2000 NCTM number standard for grades 6-8. Specifically, we find that CMP students are not expected to compute fluently, flexibly and efficiently with fractions, decimals and percents as late as 8^{th} grade. Standard algorithms for computations with fractions (e.g. $\frac{a}{b} \times \frac{b}{c} = \frac{a}{c}, \quad \frac{a}{b} \div \frac{a}{c} = \frac{c}{b}$) are often not used. We understand that the developers of CMP are aware of the absence of material on division of fractions and probably will correct this in the next edition. Conversion of fractions to decimals is discussed only in simple cases such as for fractions with denominators of ten, and CMP lacks a discussion of repeated decimals. A discussion of long division is also missing. Such a discussion could make the conversion of fractions like $\frac{1}{7}$ to decimal form a simple procedure and would tie in with a discussion of rational numbers and repeated decimals. Long division is also a basis for the division of algebraic polynomials that students will see in high school. Multiplication of fractions is discussed in 7^{th} grade but mostly in simple cases. This is an area where multiplication algorithms could be exploited to solidify the concept of place value.

CMP does a good job of helping students discover the mathematical connections and patterns in the algebra strand, but falls short in a follow-through with more substantial statements, generalizations, formulas or algorithms. For example, in *Growing, Growing, Growing* exponents are discussed, but the exponential laws are not explicitly written down for the students even after they are discovered. In one exercise students discover that $2^6 = (2^2)^3$, but they need more practice to reach the generalization that $(a^n)^m = a^{nm}$. There is no discussion of negative and fractional exponents except when students explore exponential functions using graphing calculators. As a result, students miss an opportunity to revisit square roots and cube roots. In the 8^{th} grade unit *Frogs, Fleas and Painted Cubes*, students are required to be able to recognize that the same equation can be modeled in more than one way, but CMP misses the opportunity to discuss the quadratic formula or the process of completing the square.

Many mathematicians and educators believe that when using a curriculum that relies on discovery learning, such as CMP, teachers must understand the material even better than when teaching from a more traditional curriculum. Moreover, since students often work in pairs or in groups, teachers must be effective in establishing a classroom where all students participate in the mathematical work. Also, in order for students to effectively discover the mathematics, more time needs to be devoted to the lessons than in a traditional curriculum. The recommended minimum of 45 minute-long classes seems insufficient.

In conclusion, CMP corresponds well to the 2000 NCTM Principles and Standards, with the notable exception of the number standard. We feel that CMP's overwhelming emphasis on conceptual development neglects standard computational methods and techniques. In our opinion, concepts and computations often positively reinforce one another. While we understand that CMP seems to be motivated by the criticism that traditional curricula produce students that can compute but lack conceptual understanding, there is a danger here of producing students with conceptual understanding but limited computational skills. CMP admits that "because the curriculum does not emphasize arithmetic computations done by hand, some CMP students may not do as well on parts of the standardized tests assessing computational skills as students in classes that spend most of their time on practicing such skills." This statement implies we have still not achieved a balance between teaching fundamental ideas and computational methods.

2.3 The Mathematics in Context Curriculum

The study team reviewed the Mathematics in Context (MIC) middle school curriculum, published by the Encyclopaedia Britannica Educational Corporation and funded, in part, by the National Science Foundation (NSF). MIC was developed by the Wisconsin Center for Educational Research, at the University of Wisconsin-Madison, and by the Freudenthal Institute of the University of Utrecht, the Netherlands. Information on MIC can be found at its website (http://www.ebmic.com/ebec/index.htm). The curriculum was developed to be in line with the pedagogy and content in the 1989 National Council of Teachers of Mathematics (NCTM) standards, namely the *Curriculum and Evaluation Standards for School Mathematics*, the *Professional Standards for Teaching Mathematics*, and the *Assessment Standards for School Mathematics.*

An important factor to keep in mind when reading this curriculum is its division into four parts, one each for grades *(5/6)*, *(6/7)*, *(7/8)* and *(8/9)*. This allows for some flexibility in how it would be used in a school, but must be kept in mind when comparing MIC to a standard three year curriculum. A sample curriculum ("Plan B") is included for those using Mathematics in Context as a three-year, rather than as a four-year, program. (In assessing MIC, we first look for evidence within the Plan B materials. If we need to extend the search beyond these materials, we note this in the report.) Each of the four grade levels is further divided into 10 units, contained in a separate book per unit. Each unit belongs to one of four topical strands: algebra, geometry, statistics, or number. The books are color-coded by grade-level and by strand, making organization simple. There are both student books and Teacher Guides for each unit. We reviewed only the Teachers' Guides; however, since the Teachers' Guides contain the student pages, we know both what the instructor and what the student would see. A notable exception to this is that the Teachers' Guides lacked the "Try This!" activities, a flaw which should definitely be corrected in future editions. The Teachers' Guides also contain assessments, blackline masters, and a form letter to parents describing the purpose of each unit and how they can best support their child's mathematical development at home. An overview of the MIC curriculum is given in the Teacher Resource and Implementation Guide.

Pedagogically, the program should be very simple - in theory - for teachers, even novices, to implement. Since the curriculum is broken up into units composed of bite-size lessons, and since the teacher is provided with a clear explanation of how each unit fits into the curriculum's overall scheme, there can be little confusion about how to structure the program. Alongside each lesson are comments about the underlying mathematical concepts in the lesson ("About the Mathematics"), as well as how to plan and to actually teach the lesson. A nice feature is that these comments occur in the margins of the Teachers' Guides, the bulk of which are occupied by replicas of the actual student pages, which should make the books very easy to use for educators. On the other hand, these comments often contain some useful mathematical facts and language that could be, but most likely wouldn't be, communicated to the students; in particular, high-end students could benefit from these insights if they were available to them. In addition, the lack of a glossary hides mathematical terminology from the students, a language which they should be beginning to negotiate by the middle grades. Exposure to the precise terminology of mathematics is crucial to students at this stage, not only as a means of exemplifying the rigor of mathematics, but as a way to communicate their discoveries and hypotheses in a common language, rather than the idiosyncratic terms that a particular student or class may develop. The glossaries in the Teachers' Guides should be made part of the student books. This would also be helpful to parents trying to help their child with his or her homework.

A problem, created as a by-product of the teacher-friendly units, is that the curriculum lacks coherence. The units are designed as stand-alone topics, or at best as continuations in a particular strand. Previously studied topics are not integrated well, and interaction between the strands is minimal. Students will come out of the program seeing mathematics as coming in unit-size chunks, rather than as part of a larger whole. This could easily be ameliorated, though, through exercises that emphasize connection across the curriculum.

Our subjective evaluation of the Mathematics in Context curriculum is mixed. The curriculum is very good at teaching basic conceptual points: what statistics are and how they can be used or misused; how different representations can be used to answer different types of problems effectively; what multiplication and division mean; how fractions, decimals, and percents are related; and so on. The students often develop these ideas to some extent on their own. Many of the lessons are to be used for individual or small group explorations and then brought into focus by the teacher; certainly students will get a chance to feel that they really own the material. Additionally, a "Letter to the Student" opens each book, explaining to the student what they can expect to gain from the unit, imparting a sense of educational control and responsibility to the student (as well as his or her parents). The work is almost always tied to some real context that the students can understand, and efforts are made to give examples that would interest students regardless of their gender, race, or class. However, it is our impression that the writing, examples, and pictures may be aiming a little low for students of this age group.

Our central criticism of the Mathematics in Context curriculum concerns its failure to meet elements of the 2000 NCTM number standards. Because MIC is so fixated on conceptual underpinnings, computational methods and efficiency are slighted. Formal algorithms for, say, dividing fractions are neither taught nor discovered by the students. The students are presented with the simplest numerical problems, and harder calculations are performed using calculators. Students would come out of the curriculum very calculator-dependent, and we find it hard to believe that students could divide 3.67 by .02 efficiently, even if they could explain what it meant, in abstraction, to perform that division. To us, this represents a radical change from the old "drill-and-kill" curricula, in which calculation itself was over-emphasized. The pendulum has, apparently, swung to the other side, and we feel that a return to some middle ground emphasizing both conceptual knowledge and computational efficiency is warranted. In addition, much of the time spent on the number strand is aimed too low. Students should have an understanding of equivalent fractions, decimals, and percents before the middle grades; multiplication of a decimal by a power of ten should be accomplished before grades *(6/7)*. Exponents are only used with base ten. Thus, teachers using Mathematics in Context would need to greatly augment the number strand.

There are more limited difficulties with the geometry and measurement strands. Discussion of cones and cylinders should be more prominent, and there is no discussion of density. Much more depth is needed in coordinate geometry. On the stronger side is the algebra strand, which we found fully met the standards.

Another concern, touched on above, is that high-end students may not find this curriculum very challenging or stimulating. The language and examples are fairly simple, so that such a student may feel she is being talked down to. Options for the teacher to scale-up the curriculum for advanced students are limited. The teacher could set up more advanced units for the student, which is the obvious intent of the *(5/6)*, *(6/7)*, *(7/8)*, *(8/9)* construction of the curriculum; yet even at a more advanced conceptual level, the student would still only be exposed to simple examples. Most likely the teacher would have to reach outside the curriculum to find supplemental materials.

In fact, the level of mathematics in MIC is often too low for students, particularly in the *(8/9)* books. For example, in "Going the Distance," the last unit in the geometry sequence, formulae for the area of a circle, the area of a general triangle, and the circumference of a circle are developed, as is the Pythagorean Theorem. (Some of this material is being covered for the second time, since it is worked with in "Reallotment," a *(6/7)* unit.) In "Graphing Equations," the first of the units in the *(8/9)* algebra strand, considerable time is spent graphing lines, given a linear equation. These topics are all covered by sixth grade in the Everyday Mathematics curriculum.

The deliberate emphasis of the program is on encouraging students to think about and to discuss mathematics, particularly in a problem-solving context, rather than on a direct transmission of a set body of material. Students are expected to be, and encouraged to be, active participants in developing their mathematical knowledge, rather than passive recipients of information. The underlying idea is that students today will need to use mathematics to solve real world problems; thus, presenting mathematics in a realistic setting will be both useful and interesting to students. It is hoped that this problem-solving, conceptual approach will provide students with a deeper understanding of middle school mathematics, as opposed to the superficial background they may have obtained from most curricula used in the United States in the past. As opposed to the lecture-drill format, the structure of the lessons in MIC emphasizes discovery, and teachers using MIC should be comfortable with this mode of instruction and the significant amount of time it requires. Additionally, such a curriculum will require teachers to have a deeper understanding of the mathematics than when using a more traditional approach.

In conclusion, Mathematics in Context corresponds well to the 2000 NCTM Principles and Standards, with the notable exception of the number standard. Going a bit beyond these standards, however, we feel that MIC places an over-emphasis on learning by discovery, which prevents the curriculum from reaching more advanced material by grades *(8/9)*. The curriculum, while covering basic concepts well, does not reach the level of mathematics we would hope to see by the end of the middle grades. Advanced students, in particular, would feel unchallenged.

3 Comparisons with the 2000 NCTM Standards

3.1 Methodology

In this chapter, we use the 2000 NCTM Standards as a guide to gather information about the three curricula. We then use this information (evidence) to score each curriculum on how well it compares with these standards. A brief summary for each curriculum is given for each of the ten overarching standards. In Chapter 4, we use the 2000 NCTM Principles to compare each curriculum in a similar way with the six principles. We now describe the source of our questions, the meaning of our scoring system, the criteria we use in presenting the evidence, and the purpose of the brief summaries.

The Questions:

The questions for each of the ten overarching 2000 NCTM Standards were taken verbatim from the "bullets" listed as the *Expectations* for what grades 6-8 students should be able to do. These can be found in the *Principles and Standards for School Mathematics* section of the NCTM web page, http://www.nctm.org. We formulated each bullet as a question, and as a result, the number of questions we examine is not the same for each overarching standard. The overarching standards and the number of questions used for each are: Number(14), Algebra(9), Geometry(12), Measurement(9), Data Analysis and Probability(10), Problem Solving(4), Reasoning and Proof(4), Communication(4), Connections(3), and Representation(3).

Most of the questions for each of the six 2000 NCTM Principles were taken from the *Guidebook to Examine School Curricula – TIMMS as a Starting Point to Examine Curricula, Attaining Excellence Report*, US Dept. of Education, Office of Educational Research and Improvement. This report is available at http://timss.enc.org/topics/timss/kit. The project team composed other questions after reading in detail the *Principles for School Mathematics* document, available at the NCTM site. These questions were posed in the same format as those from the *Guidebook* above. These principles and the number of questions used for each are: Equity(4), Curriculum(3), Teaching(1), Learning(2), Assessment(2), and Technology(1).

The reasons we chose the 2000 NCTM Principles and Standards as our guide are threefold. First, these standards are the most recent version available from NCTM. The second reason is the availability and knowledge of these standards by the various communities that may benefit from our report. The third reason, as we mentioned in the Introduction (p. 2), is that these standards represent a tremendous amount of thoughtful effort from a community with a wide variety of expertise, and we can comfortably subscribe to them, whether or not we agree with them in every detail.

The Evidence: (Only included in the full report.)

Each book involved in the study was examined and references gathered to compare the curricular materials with the standard. In reporting this evidence, we seek to meet the following two goals:

1. List as much evidence as necessary to support our conclusion about how fully the standard is met.

2. List evidence for each grade to show how well the curriculum builds up the concepts across the grades to meet the standard. For CMP, our goal is to list evidence at all three grades *(6)*, *(7)*, and *(8)* for each question. For MIC, where the grades were *(5/6)*, *(6/7)*, *(7/8)*, and *(8/9)* our goal is to list evidence that includes *(5/6)* or *(6/7)*, *(6/7)* or *(7/8)*, and *(7/8)* or *(8/9)*. For Singapore, our goal is to list evidence that includes *(6A)* (or *(6B)*), *(SL1)*, and *(SL2)*.

With these goals in mind, it is important to know the following before reading the report:

1. For the Number, Algebra, Geometry, Measurement, and Data and Probability Standards, we are evaluating whether the entire middle grades curriculum meets the standard. Every grade level does not necessarily have to address every question. The 2000 NCTM Principles and Standards document makes this point clear. If the standard is met before 6th grade, we note that as well.

2. For the Problem Solving, Reasoning and Proof, Communication, Connections, and Representation Standards, and the six Principles, we are evaluating whether the standard (or principle) is frequently met. The 2000 NCTM Principles and Standards document also makes this point clear. The expectation is that evidence of the standard (or principle) should be seen frequently in each grade.

3. If evidence is not supplied for a grade, it is because we were not able to find any.

4. In answering different questions, we may supply different amounts of evidence. It is possible that one of the first five standards is met totally in one book at one grade level. If this happens, we still make an effort to report what has happened at grades below or above to show the entire conceptual story across the grades in the study. On the other hand, if the materials only meet part of the standard, we still report the conceptual story that is found in the materials.

The Scoring:

The questions relating to the ten NCTM overarching standards were scored with the numbers 3, 2, 1, and 0. The meaning of each is given below.

- **Score of 3: The standard is fully met.** By the end of grade *(8)* for CMP, *(8/9)* for MIC, and *(SL2)* for Singapore, the answer to the question is an unqualified yes.

- **Score of 2: The standard is adequately met.** By the end of grade *(8)* for CMP, *(8/9)* for MIC, and *(SL2)* for Singapore, the answer to the question is not an unqualified yes. Evidence has been found for meeting parts of the standard. A judgment has been made that the curriculum's material, though not fully meeting the standard, is adequate for use.

- **Score of 1: The standard is not adequately met.** By the end of grade *(8)* for CMP, *(8/9)* for MIC, and *(SL2)* for Singapore, the answer to the question is not an unqualified yes (same as for a score of 2). Although evidence has been found for meeting parts of the standard, a judgment has been made that the curriculum's material related to this standard is not adequate for use.

- **Score of 0: The standard is not met.** By the end of grade *(8)* for CMP, *(8/9)* for MIC, and *(SL2)* for Singapore, no evidence was found for meeting the standard.

The questions relating to the six 2000 NCTM Principles are also scored with the numbers 3, 2, 1, and 0. The meaning for each of these scores is given directly below each question. The score of 2 also carries the connotation that the material does not fully meet the standard, but is adequate for use, while a score of 1 implies that the material related to this standard is not adequate for use.

The Summaries:

After each overarching standard and the principles are scored, we tabulate the scores for each question for each curriculum. We then give a short written summary for each curriculum. The purposes of this written summary are twofold:

1. To highlight the main deficiencies that led to the scores less than 3.

2. To point out, in some cases, facts worth mentioning that the evidence revealed, but that the scoring did not reflect.

Note: In what follows for CMP, IV1 refers to Investigation 1, IV2 to Investigation 2, etc.

3.2 Number Standard

3.2.1 Number Standard Question 1.

Does the curriculum enable all students to work flexibly with fractions, decimals, and percents to solve problems?

3.2.2 Number Standard Question 2.

Does the curriculum enable all students to compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line?

3.2.3 Number Standard Question 3.

Does the curriculum enable all students to develop meaning for percents greater than 100 and less than 1?

3.2.4 Number Standard Question 4.

Does the curriculum enable all students to understand and use ratios and proportions to represent quantitative relationships?

3.2.5 Number Standard Question 5.

Does the curriculum enable all students to develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation?

3.2.6 Number Standard Question 6.

Does the curriculum enable all students to use factors, multiples, prime factorization, and relatively prime numbers to solve problems?

3.2.7 Number Standard Question 7.

Does the curriculum enable all students to develop meaning for integers and represent and compare quantities with them?

3.2.8 Number Standard Question 8.

Does the curriculum enable all students to understand meaning and effects of arithmetic operations with fractions, decimals, and integers?

3.2.9 Number Standard Question 9.

Does the curriculum enable all students to use the associative, commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals?

3.2.10 Number Standard Question 10.

Does the curriculum enable all students to understand the use of inverse relationships of addition and subtraction, multiplication and division, and squaring and finding the square roots to simplify computations and solve problems?

3.2.11 Number Standard Question 11.

Does the curriculum enable all students to select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply selected methods?

3.2.12 Number Standard Question 12.

Does the curriculum enable all students to develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use?

3.2.13 Number Standard Question 13.

Does the curriculum enable all students to develop and use strategies to estimate the results of rational number computations and judge reasonableness of results?

3.2.14 Number Standard Question 14.

Does the curriculum enable all students to develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios?

3.2.15 Number Standard Summary

Singapore:

The scores in Table 1. show that the curriculum fully meets eleven of the standards, but does not adequately meet three of them. The lower scores were given because there are minimal requirements in the curriculum for students to select, develop, and analyze methods and algorithms. The evidence points to several other issues worth mentioning that the scoring did not reflect.

- No calculators are used in *(6A)* or *(6B)*.
- Negative numbers are not studied until *(SL1)*.

CMP:

The scores in Table 1. show that the curriculum fully meets ten of the standards, adequately meets one of them, and does not adequately meet three of them. The lower scores were given because the curriculum does not enable students to adequately work fluently with fractions, decimals, percents, and integers at the mathematical level expected for grades 6-8. Furthermore, the curriculum does not address the associative law. The evidence points to several other issues worth mentioning that the scoring did not reflect.

- Negative numbers are not studied until the 7th grade.

Question	Singapore	CMP	Math-in-Context
Number 1.	3	2	2
Number 2.	3	3	3
Number 3.	3	3	3
Number 4.	3	3	3
Number 5.	3	3	2
Number 6.	3	3	2
Number 7.	3	3	3
Number 8.	3	1	1
Number 9.	3	1	3
Number 10.	3	3	3
Number 11.	1	3	3
Number 12.	1	1	1
Number 13.	3	3	3
Number 14.	1	3	3

Table 1: Summary of NCTM Number and Operations Standard Results

- The associative, distributive, and commutative laws are not encountered until the 8th grade. Moreover, these concepts do not build up from early arithmetic, which would make the algebraic notions stronger.
- Proportionality and ratios are not studied until 7th grade.

MIC:

The scores in Table 1. show that the curriculum fully meets nine of the standards, adequately meets three of them, and does not adequately meet two of them. The lower scores were given because the curriculum does not enable students to adequately work fluently with fractions, decimals, percents, and integers at the mathematical level expected for grades 6-8. There is also no mention of finding the common factors of two numbers anywhere in the curriculum. The evidence points to several other issues worth mentioning that the scoring did not reflect.

- Only at the *(8/9)* level in the *Reflections on Number* unit do students see an algorithm for multiplying integers with multiple digits and that unit is not included in Plan B. It is conceivable, therefore, that students can complete a middle years mathematics program without having seen such an algorithm and having had to *analyze* why it works, or having seen its utility when numbers are not simple. This unit is also the only one in the curriculum where the topic of factors, multiples, and prime factorization of whole numbers are addressed.

- The names of the three fundamental mathematical laws (associative, communicative, distributive) do not appear in the student books.

3.3 Algebra Standard

3.3.1 Algebra Standard Question 1.

Does the curriculum enable all students to represent, analyze, and generalize a variety of patterns with tables, graphs, words, and when possible symbolic rules?

3.3.2 Algebra Standard Question 2.

Does the curriculum enable all students to relate and compare different forms of representation for a relationship?

3.3.3 Algebra Standard Question 3.

Does the curriculum enable all students to identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations?

3.3.4 Algebra Standard Question 4.

Does the curriculum enable all students to develop an initial conceptual understanding of different uses of variables?

3.3.5 Algebra Standard Question 5.

Does the curriculum enable all students to explore relationships between symbolic expressions and graphs of lines, paying particular attention to meaning of slope and intercept?

3.3.6 Algebra Standard Question 6.

Does the curriculum enable all students to use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships?

3.3.7 Algebra Standard Question 7.

Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations?

3.3.8 Algebra Standard Question 8.

Does the curriculum enable all students to model and solve contextualized problems using various representations such as graphs, tables, and equations?

3.3.9 Algebra Standard Question 9.

Does the curriculum enable all students to use graphs to analyze the nature of changes in quantities in linear relationships?

3.3.10 Algebra Standard Summary

Question	Singapore	CMP	Math-in-Context
Algebra 1.	3	3	3
Algebra 2.	3	3	3
Algebra 3.	1	3	3
Algebra 4.	3	3	3
Algebra 5.	3	3	3
Algebra 6.	3	3	3
Algebra 7.	3	3	3
Algebra 8.	3	3	3
Algebra 9.	3	3	3

Table 2: Summary of NCTM Algebra Standard Results

Singapore:

The scores in Table 2. show that the curriculum fully meets eight of the nine algebra standards and does not adequately meet one of them. The lower score was given because students do not identify functions as linear or nonlinear from tabular data. The evidence points to several other issues worth mentioning that the scoring did not reflect.

- All the algebra in *(6A)* and *(6B)* is with expressions and not equations. The first instance of modeling a word problem with an equation appears in *(SL1)*.
- Analyzing linear graphs starts in *(5B)*. This topic is not continued again until *(SL2)*, where both linear and quadratic graphs are studied.
- The concepts of slope and intercept start in *(SL1)* and continue into *(SL3)* (the text corresponding to 9th grade).

CMP:

The scores in Table 2. show that the curriculum fully meets all nine algebra standards. The evidence points to one issue worth mentioning that the scoring did not reflect.

- CMP only includes very minimal algebraic material in its 6th grade curriculum. (Reported in questions 1 and 9.)

MIC:

The scores in Table 2. show that the curriculum fully meets all nine algebra standards.

3.4 Geometry Standard

3.4.1 Geometry Standard Question 1.

Does the curriculum enable all students to precisely describe, classify, and understand relationships among types of 2D and 3D objects (e.g. angles, triangles, quadrilaterals, cyclinder, cones) using their defining properties?

3.4.2 Geometry Standard Question 2.

Does the curriculum enable all students to understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects?

3.4.3 Geometry Standard Question 3.

Does the curriculum enable all students to create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity and Pythagorean relationship?

3.4.4 Geometry Standard Question 4.

Does the curriculum enable all students to use coordinate geometry to represent and examine the properties of geometric shapes?

3.4.5 Geometry Standard Question 5.

Does the curriculum enable all students to use coordinate geometry to examine special geometric shapes, such as regular polygons or those with pairs of parallel or perpendicular sides?

3.4.6 Geometry Standard Question 6.

Does the curriculum enable all students to describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling?

3.4.7 Geometry Standard Question 7.

Does the curriculum enable all students to examine the congruence, similarity, and line of rotational symmetry of objects using transformations?

3.4.8 Geometry Standard Question 8.

Does the curriculum enable all students to draw geometric objects with specified properties, such as side lengths or angle measures?

3.4.9 Geometry Standard Question 9.

Does the curriculum enable all students to use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involving surface area and volume?

3.4.10 Geometry Standard Question 10.

Does the curriculum enable all students to use visual tools such as networks to represent and solve problems?

3.4.11 Geometry Standard Question 11.

Does the curriculum enable all students to use geometric models to represent and explain numerical and algebraic relationships?

3.4.12 Geometry Standard Question 12.

Does the curriculum enable all students to recognize and apply geometric ideas and relationships in areas outside the mathematics classroom, such as art, science, and everyday life?

3.4.13 Geometry Standard Summary

Singapore:

The scores in Table 3. show that the curriculum fully meets ten of the twelve geometry standards, adequately meets one standard, and does not adequately meet one standard. The lower scores were given because students do not do inductive reasoning related to geometry, and they only work minimally with visual tools such as networks. The evidence points to another issue worth mentioning that the scoring did not reflect.

- The curriculum does not make explicit that students should critique the arguments of others.

CMP:

The scores in Table 3. show that the curriculum fully meets eleven of the twelve geometry standards and adequately meets one standard. The lower score was given because the construction of objects to specification was only done for very simple objects, such as parallelograms and rectangles. The evidence points to other issues worth mentioning that the scoring did not reflect.

Question	Singapore	CMP	Math-in-Context
Geometry 1.	3	3	2
Geometry 2.	3	3	3
Geometry 3.	2	3	3
Geometry 4.	3	3	1
Geometry 5.	3	3	0
Geometry 6.	3	3	3
Geometry 7.	3	3	3
Geometry 8.	3	2	3
Geometry 9.	3	3	3
Geometry 10.	1	3	3
Geometry 11.	3	3	3
Geometry 12.	3	3	3

Table 3: Summary of NCTM Geometry Standard Results

- We did not find evidence that students create or critique inductive or deductive arguments in 6th grade.
- Geometric models are not used to represent and explain numerical and algebraic relationships until 8th grade.

MIC:

The scores in Table 3. show that the curriculum fully meets nine of the twelve geometry standards, adequately meets one standard, does not adequately meet one standard, and does not meet one standard. The lower scores were given because students work minimally with coordinate geometry and with cylinders and cones. The evidence points to another issue worth mentioning that the scoring did not reflect.

- The unit *(8/9) Going the Distance* is the only unit in MIC that addresses the Pythagorean relationship. This unit is not included in Plan B, the recommended books for a three year middle-grades program.

3.5 Measurement Standard

3.5.1 Measurement Standard Question 1.

Does the curriculum enable all students to understand both metric and customary systems?

3.5.2 Measurement Standard Question 2.

Does the curriculum enable all students to understand relationships among units and convert from one unit to another within the same system?

3.5.3 Measurement Standard Question 3.

Does the curriculum enable all students to understand, select, and use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume?

3.5.4 Measurement Standard Question 4.

Does the curriculum enable all students to use common benchmarks to select appropriate methods for estimating measurements?

3.5.5 Measurement Standard Question 5.

Does the curriculum enable all students to select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision?

3.5.6 Measurement Standard Question 6.

Does the curriculum enable all students to develop and use formulas to determine the circumference of circles and areas of triangles, parallelograms, trapezoids, circles, and develop strategies to find areas of more complex shapes?

3.5.7 Measurement Standard Question 7.

Does the curriculum enable all students to develop strategies to determine the surface area and volume of selected prisms, pyramids, and cylinders?

3.5.8 Measurement Standard Question 8.

Does the curriculum enable all students to solve problems involving scale factors, using ratio and proportion?

3.5.9 Measurement Standard Question 9.

Does the curriculum enable all students to solve simple problems involving rates and derived measurements for such attributes as velocity and density?

3.5.10 Measurement Standard Summary

Singapore:

Question	Singapore	CMP	Math-in-Context
Measurement 1.	3	3	3
Measurement 2.	3	1	3
Measurement 3.	3	3	3
Measurement 4.	0	3	3
Measurement 5.	3	3	3
Measurement 6.	3	3	3
Measurement 7.	3	3	2
Measurement 8.	3	3	3
Measurement 9.	3	3	2

Table 4: Summary of NCTM Measurement Standard Results

The scores in Table 4. show that the curriculum fully meets eight of the nine measurement standards, and does not meet one standard. The lower score was given because students do not use common-sense benchmarks to estimate or measure objects.

CMP:

The scores in Table 4. show that the curriculum fully meets eight of the nine measurement standards, and does not adequately meet one standard. The lower score was given because students work minimally with measurement conversions within the same measurement system. The evidence points to another issue worth mentioning that the scoring did not reflect.

- In the problems students do, density always refers to population density (which has units of mass per unit area). No examples from physics were included where density is interpreted as mass per unit volume.

MIC:

The scores in Table 4. show that the curriculum fully meets seven of the nine measurement standards, and adequately meets two standards. The lower scores were given because students do not work with volumes of cylinders or do calculations involving density.

3.6 Data and Probability Standard

3.6.1 Data and Probability Standard Question 1.

Does the curriculum enable all students to formulate questions, design studies and collect data about a characteristic shared by two populations or different characteristics within one population?

3.6.2 Data and Probability Standard Question 2.

Does the curriculum enable all students to select, create, and use appropriate graphical representations of data, including histograms, box plots, and scatterplots?

3.6.3 Data and Probability Standard Question 3.

Does the curriculum enable all students to find, use, and interpret measures of center and spread, including mean and interquartile range?

3.6.4 Data and Probability Standard Question 4.

Does the curriculum enable all students to discuss and understand the correspondence between data sets and their graphical representations, especially histograms, stem-and-leaf plots, box plots, and scatterplots?

3.6.5 Data and Probability Standard Question 5.

Does the curriculum enable all students to use observations about differences between two or more samples to make conjectures about the populations from which samples were taken?

3.6.6 Data and Probability Standard Question 6.

Does the curriculum enable all students to make conjectures about possible relationships between two characteristics of a sample on the basis of scatterplots of the data and approximate lines of fit?

3.6.7 Data and Probability Standard Question 7.

Does the curriculum enable all students to use conjectures to formulate new questions and plan new studies to answer them?

3.6.8 Data and Probability Standard Question 8.

Does the curriculum enable all students to understand and use appropriate terminology to describe complementary and mutually exclusive events?

3.6.9 Data and Probability Standard Question 9.

Does the curriculum enable all students to use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations?

3.6.10 Data and Probability Standard Question 10.

Does the curriculum enable all students to compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area models?

3.6.11 Data and Probability Standard Summary

Question	Singapore	CMP	Math-in-Context
Data Anal. and Prob. 1.	3	3	3
Data Anal. and Prob. 2.	2	3	3
Data Anal. and Prob. 3.	2	3	3
Data Anal. and Prob. 4.	2	3	3
Data Anal. and Prob. 5.	0	3	3
Data Anal. and Prob. 6.	3	3	3
Data Anal. and Prob. 7.	0	0	0
Data Anal. and Prob. 8.	0	2	3
Data Anal. and Prob. 9.	0	3	3
Data Anal. and Prob. 10.	0	3	3

Table 5: Summary of NCTM Data Anal. and Prob. Standard Results

Singapore:

The scores in Table 5. show that the curriculum fully meets two of the standards, adequately meets three of them, and does not meet five of them. The lowest scores were given because there is a lack of any probability in the curriculum for the middle grades. (Probability begins in *(SL4)*.) The reasons for the other low scores are the lack of conjecturing about relationships between two characteristics of a sample, the lack of making conjectures to formulate new questions and plan new studies, and the lack of work with box plots, interquartile range, or stem and leaf plots. The evidence points to other issues worth mentioning that the scoring did not reflect:

- The only statistics in the *(6A)* and *(6B)* books involves reading data from pie charts.
- There is no statistics in *(SL1)*.
- The curriculum does not make explicit that students should critique the arguments of others.

CMP:

The scores in Table 5. show that the curriculum fully meets eight of the ten standards, adequately meets one standard, and does not meet one standard. The lower scores were given because students do not design further studies after an initial study has been completed and do not use the vocabulary of complementary and mutually exclusive events. The evidence points to another issue worth mentioning that the scoring did not reflect:

- With the exception of the capture-tag-recapture technique in the 7th grade *Comparing and Scaling* unit, there is no statistics in 7th grade CMP.

MIC:

The scores in Table 5. show that the curriculum fully meets nine of the ten standards, and does not meet one standard. The lower score was given because students do not ask new questions and design new studies after an initial study has been completed.

3.7 Problem Solving Standard

3.7.1 Problem Solving Standard Question 1.

Does the curriculum enable all students to build new math knowledge through problem solving?

3.7.2 Problem Solving Standard Question 2.

Does the curriculum enable all students to solve problems that arise in math and in other contexts?

3.7.3 Problem Solving Standard Question 3.

Does the curriculum enable all students to apply and adapt a variety of appropriate strategies to solve problems?

3.7.4 Problem Solving Standard Question 4.

Does the curriculum enable all students to monitor and reflect on the process of mathematical problem solving?

3.7.5 Problem Solving Standard Summary

Singapore:

The scores in Table 6. show that the curriculum fully meets one of the standards, meets two adequately, and does not meet the fourth. The lower scores were given because the curriculum does not frequently include contextual problems, does not provide

Question	Singapore	CMP	Math-in-Context
Problem Solving 1.	3	3	3
Problem Solving 2.	2	3	3
Problem Solving 3.	2	3	3
Problem Solving 4.	0	3	3

Table 6: Summary of NCTM Problem Solving Standard Results

problems where students must adapt the strategies given in the book, and does not explicitly ask students to reflect on the process of mathematical problem solving.

CMP:

The scores in the Table 6. show that the curriculum fully meets all four of the standards.

MIC:

The scores in the Table 6. show that the curriculum fully meets all four of the standards.

3.8 Reasoning and Proof Standard

3.8.1 Reasoning Standard Question 1.

Does the curriculum enable all students to recognize reasoning and proof as fundamental aspects of math?

3.8.2 Reasoning Standard Question 2.

Does the curriculum enable all students to make and investigate math conjectures?

3.8.3 Reasoning Standard Question 3.

Does the curriculum enable all students to develop and evaluate math arguments and proofs?

3.8.4 Reasoning Standard Question 4.

Does the curriculum enable all students to select and use various types of reasoning and methods of proof?

Question	Singapore	CMP	Math-in-Context
Reason and Proof 1.	1	3	3
Reason and Proof 2.	1	3	3
Reason and Proof 3.	1	3	3
Reason and Proof 4.	3	3	3

Table 7: Summary of NCTM Reason and Proof Standard Results

3.8.5 Reasoning and Proof Standard Summary

Singapore:

The scores in Table 7. show that the curriculum fully meets one standard and does not adequately meet three of them. The lower scores were given because the curriculum does not explicitly encourage students on a regular basis to explain their reasoning in the problems or assessments they do, does not ask students to frequently make and investigate their own mathematical conjectures, and does not explicitly ask students to evaluate the mathematical arguments of others. The evidence points to another issue worth mentioning that the scoring did not reflect.

- Students do both inductive and deductive reasoning and select between strategies for inductive reasoning. We note that they do not encounter problems that require them to select between inductive and deductive reasoning.

CMP:

The scores in Table 7. show that the curriculum fully meets all four of the standards.

MIC:

The scores in Table 7. show that the curriculum fully meets all four of the standards.

3.9 Communication Standard

3.9.1 Communication Standard Question 1.

Does the curriculum enable all students to organize and consolidate their math thinking through both written and oral communication?

3.9.2 Communication Standard Question 2.

Does the curriculum enable all students to communicate math thinking coherently and clearly to peers, teachers, and others? (both oral and written)

3.9.3 Communication Standard Question 3.

Does the curriculum enable all students to analyze and evaluate math thinking and strategies of others?

3.9.4 Communication Standard Question 4.

Does the curriculum enable all students to use the language of math to express math ideas precisely?

3.9.5 Communication Standard Summary

Question	Singapore	CMP	Math-in-Context
Communication 1.	1	3	3
Communication 2.	1	3	3
Communication 3.	0	3	3
Communication 4.	2	3	2

Table 8: Summary of NCTM Communication Standard Results

Singapore:

The scores in Table 8. show that the curriculum adequately meets one standard, does not adequately meet two standards, and does not meet one standard. The lower scores were given because the curriculum does not adequately provide opportunities for written communication, does not explicitly encourage students to look at each other's thinking in a critical way, and does not encourage students to write about mathematics in order to use mathematical terminology.

CMP:

The scores in Table 8. show that the curriculum fully meets all four of the standards.

MIC:

The scores in Table 8. show that the curriculum fully meets three of the four standards and adequately meets one of them. The lower score was given because mathematical terminology is not prevalent in the student books.

3.10 Connection Standard

3.10.1 Connection Standard Question 1.

Does the curriculum enable all students to recognize and use connections among math ideas?

3.10.2 Connection Standard Question 2.

Does the curriculum enable all students to understand how math ideas interconnect and build to create a coherent whole?

3.10.3 Connection Standard Question 3.

Does the curriculum enable all students to recognize and apply math in contexts outside math?

3.10.4 Connection Standard Summary

Question	Singapore	CMP	Math-in-Context
Connections 1.	1	3	1
Connections 2.	1	3	1
Connections 3.	1	3	3

Table 9: Summary of NCTM Connections Standard Results

Singapore:

The scores in Table 9. show that the curriculum did not adequately meet any of these standards. The lower scores were given because the problems rarely integrate different strands of math at the same time, that these problems are very similar on a given topic, and are mainly context-free. The evidence points to several other issues worth mentioning that the scoring did not reflect.

- The *(6A)* and *(6B)* books are much better than the *(SL1)* and *(SL2)* books in pointing out to the teacher how the mathematical ideas connect.

CMP:

The scores in Table 9. above show that the curriculum fully meets all three of the standards.

MIC:

The scores in Table 9. show that the curriculum fully meets one of the three standards and does not adequately meet two of them. The lower scores were given because the connections between mathematical strands, though very explicit in the teacher books, are not reflected in student books nor in the nature of the problems students do.

3.11 Representation Standard

3.11.1 Representation Standard Question 1.

Does the curriculum enable all students to create and use representations to organize, record, and communicate mathematical ideas?

3.11.2 Representation Standard Question 2.

Does the curriculum enable all students to select, apply, and translate among mathematical representations to solve problems?

3.11.3 Representation Standard Question 3.

Does the curriculum enable all students to use representations to model and interpret physical, social, and mathematical phenomena?

3.11.4 Representation Standard Summary

Question	Singapore	CMP	Math-in-Context
Representation 1.	3	3	3
Representation 2.	3	3	3
Representation 3.	2	3	3

Table 10: Summary of NCTM Representation Standard Results

Singapore:

The scores in Table 10. show that the curriculum fully meets two of these standards, and adequately meets the third. The lower score was given because the curriculum does not frequently include problems that require students to model and interpret physical and social phenomena.

CMP:

The scores in Table 10. show that the curriculum fully meets all three of the standards.

MIC:

The scores in Table 10. show that the curriculum fully meets all three of the standards.

4 Comparisons with the 2000 NCTM Principles

4.1 The Equity Principle

Excellence in mathematics education requires equity-high expectations and strong support for all students. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students.

4.1.1 Equity Question 1.

Does the curriculum provide materials and suggestions to the teacher for individualizing instruction?

0	1	2	3
All students do the same tasks	Low	Medium	High number of materials and tips for individualizing

4.1.2 Equity Question 2.

Are the curriculum materials likely to be interesting, engaging, and effective for girls and boys?

0	1	2	3
No sensitivity to gender issues	Low	Medium	High sensitivity to gender issues

4.1.3 Equity Question 3.

Are the curriculum materials likely to be interesting, engaging, and effective for underrepresented and underserved students (e.g., ethnic, rural, with disabilities)?

0	1	2	3
No sensitivity to underrepresented and underserved students	Low	Medium	High sensitivity to underrepresented and underserved students

4.1.4 Equity Question 4.

Are the curriculum materials likely to be interesting, engaging, and effective for mathematically capable students?

0	1	2	3
No	Low	Medium	Highly interesting,

Level too low even with the extension materials	engaging, and effective without any extension materials

4.2 The Curriculum Principle

A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades. The interconnections between the mathematics strands should be displayed prominently in the curriculum and in instructional materials and lessons. A coherent curriculum effectively organizes and integrates important mathematical ideas so that students can see how the ideas build on, or connect with, other ideas, thus enabling them to develop new understandings and skills. The curriculum should help teachers understand the mathematics that has been studied at previous levels and what is the focus at successive levels. A well-articulated curriculum gives teachers guidance regarding important ideas and major themes and depth of study warranted at particular times and when closure is expected for particular skills or concepts.

4.2.1 Curriculum Question 1.

Is the mathematics curriculum coherent?

0	1	2	3
No	Low	Medium	Very much so

4.2.2 Curriculum Question 2.

Does the curriculum focus on important mathematics?

0	1	2	3
No	Low	Medium	Very much so

4.2.3 Curriculum Question 3.

Is the mathematics curriculum well-articulated across the grades (6-8)?

0	1	2	3
No	Low	Medium	Very much so

4.3 The Teaching Principle

Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well. To be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks.

4.3.1 Teaching Question 1.

Does the curriculum help the teacher to deeply understand the mathematics he or she needs to teach the students?

0	1	2	3
No	Low	Medium	Very much so

4.4 The Learning Principle

Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. A major goal of school mathematics is to create autonomous learners who when challenged with appropriate tasks are confident in their ability to tackle difficult problems, eager to figure things out on their own, are flexible in exploring mathematical ideas and trying alternative solution paths, and are willing to persevere.

4.4.1 Learning Question 1.

Does the curriculum promote learning with understanding?

0	1	2	3
No	Low	Medium	Very much so

4.4.2 Learning Question 2.

Does the curriculum encourage students to be autonomous learners?

0	1	2	3
No	Low	Medium	Very much so

4.5 The Assessment Principle

Assessment should support the learning of important mathematics and furnish useful information to both teachers and students. Assessment should be more than merely a test at the end of instruction to see how students perform under special conditions; rather, it should be an integral part of instruction that informs and guides teachers as they make instructional decisions.

4.5.1 Assessment Question 1.

Does the curriculum include and encourage multiple kinds of assessments (e.g. performance, formative, summative, paper-pencil, observations, portfolios, journals, student interviews, projects)?

0	1	2	3
No	Low	Medium	Very much so

4.5.2 Assessment Question 2.

Does the curriculum provide well-aligned summative assessments to judge a student's attainment?

0	1	2	3
No	Low	Medium	Very much so

4.6 The Technology Principle

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. Technology should not be used as a replacement for basic understandings and intuitions; rather it can and should be used to foster those understandings and intuitions.

4.6.1 Technology Question 1.

Does the curriculum use technology as a tool for learning and doing mathematics?

0	1	2	3
No	Low	Medium	Very much so

4.7 Summary of Alignment with NCTM Principles

Question	Singapore	CMP	Math-in-Context
Equity 1.	1	3	3
Equity 2.	1	3	3
Equity 3.	1	3	3
Equity 4.	2	2	2
Curriculum 1.	1	3	1
Curriculum 2.	3	3	3
Curriculum 3.	1	3	3
Teaching 1.	3	3	3
Learning 1.	2	3	3
Learning 2.	0	3	3
Assessment 1.	0	3	3
Assessment 2.	3	3	3
Technology 1.	2	3	2

Table 11: Summary of Alignment with the NCTM Principles

Singapore:

The scores in the table above show that the curriculum fully meets three of the thirteen principles, adequately meets three of them, does not adequately meet five of them, and does not meet two of them. The lower scores were given because of the lack of coherence, articulation, multiple forms of assessment, use of calculators in *(6A)* and *(6B)*, sensitivity to gender issues, and sensitivity to underrepresented and underserved students in the curricular materials. One of the lowest scores was due to the curriculum's lack of attention to encouraging students to learn with understanding and to be autonomous learners.

CMP:

The scores in the table above show that the curriculum fully meets twelve of the thirteen principles and adequately meets one of them. The lower score was given because the materials will have to be supplemented to keep the interest of the more capable students.

MIC:

The scores in the table above show that the curriculum fully meets ten of the thirteen principles, adequately meets two of them, and does not adequately meet one of them.

The lower scores were given because the connections between the mathematical strands is not made evident in the problems that students do, the materials will have to be supplemented to keep the interest of the more capable students, and calculators are the only technology that is frequently used in the curriculum. (In the teacher books, videos are listed that correspond to a given topic, but they appear to be optional.)

5 Direct Comparisons of the Curricula

In Chapters 3 and 4, we compared each curriculum with the 2000 NCTM Principles and Standards. That comparison was not intended to be a direct comparison of the three curricula with each other. The purpose of this chapter is to perform a direct comparison of each curriculum against each of the others, drawing upon the information we have gained from the previous chapters in conjunction with our joint expertise as mathematicians and applied mathematicians.

In the sections that follow, we compare the three curricula with each other in the areas covered by the ten overarching standards and principles. However, our discussion is not confined to those standards and principles.

5.1 Number Comparison

Ratio and Proportion:

All three curricula include good strategies for finding ratios and proportions. Singapore's strategy is called the unitary method. CMP calls it the rate table. MIC calls it the ratio table. In CMP, this topic, however, is only studied in 7th grade. MIC treats this topic as early as the *(5/6)* units and continues with it through the *(8/9)* units. Singapore's students work much more difficult problems on this topic in grades *6A* and *6B*, *SL1* and *SL2* than students using the American texts. Algebra is ultimately used to solve these problems in the Singapore texts.

The Commutative, Associative, and Distributive Laws:

CMP only introduces these important topics in 8th grade. MIC's students work with these laws in all grades *(5/6)* through *(8/9)*, but the terminology is not used in the student books. Singapore's texts, on the other hand, offer students many opportunities to work with these laws by name throughout the middle grades. Moreover, they work fluently with them. We note that the Everyday Mathematics elementary program (K-6) requires students to work with these laws in fifth grade and by name in sixth grade. In this respect, CMP aims too low.

Algorithms:

In CMP and MIC, students invent and analyze their own algorithms. The 2000 Standards say that students should compare their invented algorithms to the standard ones. We note that these standard methods can be taught with understanding and can be just as conceptual as a rate table, for instance, and have far greater utility for more complicated situations. In CMP, these standard methods will probably surface during the Summarize part of the lesson. MIC introduces the standard algorithm for multiplication of two digit integers in the *(8/9)* unit *Reflections on Number* – we feel this is much too late. Furthermore, this unit, though included in our study, is not included in Plan B, the books recommended by the publisher for a three-year middle school program. If introduced earlier and analyzed, students would have yet another conceptual way to organize and solve problems and another algorithm with which to compare their invented ones, as recommended in the standards.

The Singapore texts have a different philosophy. They present what are considered the best algorithms for the type of problem considered. Students then practice these algorithms in solving complicated problems. The *SL1* and *SL2* textbooks appear to be just books of problems. The philosophy is to provide practice, rather than opportunities to select, develop, and analyze.

Exponent Usage and Multiple Bases:

In MIC, the general case $a^b a^c = a^{(b+c)}, a \neq 0$ is only studied when the base $a = 10$. Other bases for a are used when b and c are both positive. CMP introduces students to functions of the form $y = a^x$ and $y = a^{-x}$ in the 8th grade, and uses graphing calculators to study the exponential growth and decay. On the other hand, the Singapore texts use exponents in a very general way. The laws for exponents are given explicitly in the student texts. The bases are general, the exponents are both positive and negative, and students work many problems using these laws.

On a related issue, we note that the 2000 NCTM Number Standard for grades 6-8 does not mention the use of multiple base arithmetic as a way to build up the concept of place value using exponents. This might account for the fact that MIC does not deal with powers of numbers other than 10. For example, 10^5 is related to a certain place value in the base 10 number system (100000) and 2^5 is related to the base 2 number (100000). None of the curricula we examined makes this important connection.

We also note that the Everyday Mathematics elementary curriculum works extensively with place value in grades 4 and 5.

Calculators:

CMP and MIC make use of calculators across the middle grades. Singapore makes it clear that there is no calculator usage allowed in *6A* and *6B*. Calculator usage begins in earnest in *SL1*.

Fraction, Decimals, and Percents:

CMP and MIC students do not work fluently with this topic. The calculations are on the whole too simple for these grades, especially those done toward the end of 8th grade or in the *(8/9)* books. Students are not working with general fractions to compare them by finding a common denominator. By the end of 8th grade, we feel this is a skill students should have. Instead, they use a calculator, which converts the fractions to an approximate decimal form.

CMP and MIC were designed to adhere to the 1989 NCTM Standards, which had very low standards with regard to fluency and skills involving fractions. The 2000 NCTM Standards now require a higher level of fluency and skills, which are not met by CMP and MIC. We think that the discussion in the 2000 NCTM Standards may also lead curriculum developers to aim too low in the area of computation. In particular, an algorithm for division by a fraction discussed in the 2000 NCTM Principles and Standards on p. 217 as repeated subtraction is flawed (e.g. 5 divided by 3/4 as $5-3/4-3/4-3/4-3/4-3/4-3/4=1/2$. How should one divide 1/2 by 3/4 using repeated subtraction? How can one use repeated substraction when the denominator is bigger than the numerator?), especially without the subsequent generalization to (and proof of) the more general "invert and multiply" algorithm the students will need in algebra.

The problems the Singapore students do with this topic are much more complicated than those in the American texts. In fact, the Singapore *SL1* text tells the teachers to go quickly through these topics because they were covered in depth in lower grades. We note that *SL2* does not teach these topics at all. This makes sense for the Singapore students using these texts, since students have been streamed already. Even though the topics are more complicated, Singapore students are not expected to select, develop, or analyze the algorithms they use. They simply practice them.

We note that CMP and MIC teach students "when" it is appropriate to use a decimal versus a fraction or a percent. Singapore's texts do not do this.

5.2 Algebra Comparison

MIC includes algebra throughout the middle grades. CMP has no algebra in its 6th grade curriculum. Singapore's *6A* and *6B* include algebraic expressions, but not equations. Singapore's treatment of linear graphs starts earlier in *5A* and *5B* and doesn't continue until the 8th grade book *SL2*. If Singapore's texts were used in middle schools in America, the pre-requisite work expected by *SL2* might be missing. Coordination with elementary curriculum is extremely crucial and should be kept in mind before selecting any of the three curricula.

The Algebra level in CMP and MIC appear to be almost two grade levels lower than in the Singapore materials. Division of one polynomial by another or multiplying two polynomials of order higher than one is not taught even by the 8th grade in these American curricula.

5.3 Geometry Comparison

Creating and Critiquing Arguments Related to Geometry:

This topic is missing from 6th grade CMP, but is done later in the curriculum. CMP and MIC both develop the habits of mind to critique the arguments of others, especially in the classroom setting. Singapore's curriculum, especially *SL1* and *SL2*, give the impression that students are doing problems in isolation. The curriculum does not explicitly call for this mode of operation. However, Singapore's curriculum contains a fair number of geometry proofs based on deductive reasoning that are at a much higher mathematical level than those in the American curricula. Inductive reasoning is not found in the Singapore texts in relation to geometry.

Coordinate Geometry:

CMP and Singapore both met the standards related to coordinate geometry. CMP has this topic across all grade levels. Singapore starts it in *SL1* and picks it up again for more proficiency in *SL3*. The evidence in Chapter 3 shows that MIC is very deficient in coordinate geometry.

Pythagorean Relationship:

All three curricula address this relationship in the 8th grade books. The unit *(8/9) Going the Distance* is the only unit in MIC that addresses the Pythagorean relationship. This unit is not included in Plan B, the recommended books for a three year middle-grades program. It is surprising to us that this topic is not treated in much earlier grades in all three curricula.

Geometric Models to Explain Algebra:

Singapore starts at the 5th grade and continues to use these models through 8th grade. MIC also makes use of these models from units *(6/7)* up through *(8/9)*. CMP, on the other hand, doesn't use them until 8th grade.

5.4 Measurement Comparison

Both CMP and MIC make use of common benchmarks to develop measurement skills. The Singapore curriculum does not.

The Singapore curriculum includes problems that require students to go between two different scales in the same problem. Linear and area scales are also used. CMP and MIC also include good sections on scaling. Singapore includes more work on problems involving density than CMP. MIC does not include problems involving density.

All three curricula provide problems that require conversions between the customary and metric system, but CMP does very little with conversions within the same system.

5.5 Data and Probability Comparison

The three curricula differ widely in the amount of coverage of probability and statistics in the middle grades. MIC includes these topics in all of the middle grades. CMP does not include statistics in its 7th grade books, but does have statistics in its 6th and 8th grade books. Singapore does not include statistics in its 6th or 7th grade books. With the exception of data representation, which is done in the 4th grade books, and reading pie charts in the 6th grade books, statistics begins at 8th grade in the *SL2* books. Singapore has a total lack of probability in all of Primary school (1-6) and the books *SL1*, *SL2*, and *SL3*. Probability begins at 10th grade in *SL4*. We felt this delay to be unfortunate. This issue would have to be seriously addressed if this curriculum were to be used in American classrooms.

5.6 Problem Solving Comparison

CMP and MIC are equally good at including problem solving in the curriculum. These curricula take the point of view that it is the student's job to learn how to select, adapt, and reflect on problem solving strategies. The Singapore curriculum does not require students to monitor and reflect on the process of problem solving or to adapt strategies. The philosophy seems to be that good strategies are given the students and a student's job is to apply these strategies to solve complicated, but mostly non-contextual problems.

5.7 Reasoning and Proof Comparison

CMP and MIC are both very strong in explicitly requiring students to explain their reasoning. Singapore's curriculum does not do this, especially in the *SL1* and *SL2* books. Moreover, Singapore students don't have to select strategies or make their own conjectures as often as students from CMP and MIC. We feel that MIC could provide more of a distinction between explaining one's thinking which can be flawed and providing an argument which can not be refuted.

5.8 Communication Comparison

The Singapore texts use the language of mathematics in a very rich way, but this usage doesn't carry over to the expectations for the students. They are rarely asked to explain their reasoning in writing, or to analyze or evaluate the mathematical thinking of others. The problems they work on and the assessments they do require very little writing using mathematical terminology.

MIC is the opposite of Singapore in this regard. The MIC student books avoid the precise language of mathematics. (Mathematical terminology and discussion of the general case are seen in the teacher book.) MIC students are, however, asked frequently to use written and oral communication to explain their reasoning and analyze or evaluate the mathematical thinking of others.

CMP uses the language of mathematics in the student books as well as the teacher books. CMP students are also asked frequently to use written and oral communication to explain their reasoning and analyze or evaluate the mathematical thinking of others.

5.9 Connection Comparison

A well-connected mathematics program includes problems that span multiple subject areas that incorporate different mathematical strands. The curriculum should provide the teacher with the explicit strands that are involved in the lesson. The CMP curriculum does this better than either MIC or Singapore. In CMP, such connections are made explicit for the teacher and the student. The MIC curriculum is very strong on the use of contextual problems, but not as good as CMP on making the connections explicit to the student. MIC makes the connections explicit to the teacher, but the problems the students do don't draw in these connections as well as those in CMP. We sense a philosophical difference between CMP and MIC in this regard. MIC gives the impression that the goal is to make everything available to the teacher, on the theory that if the teacher sees it, the students see it. CMP provides the connections explicitly in the ACE Problems (A is Application, C is Connection and E is Extension). Students see through the C (Connection) problems how the current topic connects back to previous work. The problems in MIC corresond to CMP's "A" and "E" problems.

Singapore's *6A* and *6B* books contain quite a lot of guidance to the teacher on how the math ideas interconnect. The *SL1* and *SL2* books present none of this. All of the Singapore books, however, only require the student to practice one strand of mathematics at a given time. The work that students do does not require the integration of different mathematical strands. Even though a mathematician can clearly see the logical progression of the mathematical ideas in these books, the connections of these ideas are not nearly as evident in the work students do as those in the CMP books.

On the other hand, the connection issue can also be interpreted as vertical connections in the internal structure of mathematics. In this regard, the exposition in Singapore's books follows a more logical progression of mathematical ideas, uses more precise mathematical language, and provides more abstraction and generalization than the exposition in the American texts. To mathematicians this logical vertical connection is very familiar and easy to appreciate, although it could be made more apparent in the problems that students do.

5.10 Representation Comparison

All three curricula are very strong in the use of representations. Singapore's students do not use representations as often in physical or social contextual problems as do the

American students. They do, however, use many representations in the mathematical problems they do.

5.11 Principles Comparison

The main deficiencies of the Singapore curriculum include its lack of coherence, articulation, multiple forms of assessment, use of calculators in *(6A)* and *(6B)*, sensitivity to gender issues, and sensitivity to underrepresented and underserved students. The most serious single deficiency, however, is its lack of attention to encouraging students to be autonomous learners. The curriculum is based on the philosophy that practicing many, many problems will lead to mathematical understanding and will produce autonomous learners.

Both CMP and MIC are more in line with the 2000 NCTM Principles than the Singapore curriculum. Both are well-articulated, provide multiple forms of assessments, use calculators throughout, are sensitive to gender issues and to underrepresented and underserved students. CMP provides problems with more coherence (connections) across mathematical strands and makes more use of technology than MIC. These American curricula take the point of view that it is the student's job to learn how to select, adapt, and reflect upon strategies for solving problems, and that by doing so, the student will develop mathematical understanding and become an autonomous learner.

Both CMP and MIC lack the mathematical level necessary to challenge the brightest students. These curricula will have to be supplemented in the number, algebra, and geometry strands in particular to meet these needs. Singapore's curriculum, on the other hand, has the mathematical level and rigor required for these students, but does not require them to rise to the level of being able to select or adapt strategies or reflect about them in writing.

6 Conclusions

In Chapters 3 and 4 we compared two American curricula, Connected Mathematics Program (CMP) and Mathematics in Context (MIC), and the Singapore material, at the middle school levels of 6th, 7th and 8th grades with the 2000 NCTM Principles and Standards. In Chapter 5 we compared these three curricula with each other. The two American curricula are actually quite similar to each other in philosophy and execution. Both were developed during the nineties to satisfy the 1989 National Council of Teachers of Mathematics (NCTM) standards, namely the *Curriculum and Evaluation Standards for School Mathematics*, the *Professional Standards for Teaching Mathematics*, and the *Assessment Standards for School Mathematics.* They represent the new thrust in American mathematical education of inquiry-based, discovery-based and student-centered learning. The American curricula strive to produce independent thinkers who can analyze problems, select appropriate tools to solve them and achieve conceptual understanding of the mathematics behind the algorithms, usually through a real-world context. Singapore mathematics, on the other hand, is more traditional in orientation and emphasizes the acquisition of proficiency in mathematical skills and teacher-directed learning. Although there are some signs that the latest Singapore curriculum is attempting to include more discovery-type tasks for students, the implementation is, at this stage, nothing more than a simple retrofit. The two American curricula, on the other hand, were designed from the beginning to reflect this new approach to learning; MIC and CMP appear to us to be much better conceived along these lines and have much better teacher support for implementing the lessons than the Singapore curriculum, especially at the the secondary grades (7th and 8th). When compared with the 2000 NCTM Principles and Standards, the Singapore curriculum scores lower than the two American texts.

This result is not surprising. The Singapore texts were designed for Singapore students to prepare them for the General Certificate of Education (GCE) examinations. The students have already been divided into different streams when they come to middle school. Therefore the curriculum targets a very specific audience at very specific student levels. The Singapore teachers are educated more uniformly at their normal school, and their continuing professional development is supported by the government at a level not achieved in the United States. Consequently, the lack of teacher support in the textbooks themselves does not appear to pose a problem for the Singapore teachers. It is interesting to point out that at the primary grades, the Singapore texts provide excellent teacher support, probably because it is thought to be necessary at these lower levels where the teachers are not mathematics or science specialists. The Singapore mathematics curriculum appears to do what it was designed to do quite well for the audience intended. The mathematical problems are more advanced than those of the two American curricula, and use the language of mathematics in a rich way. Mathematicians may find it attractive because it lays a logical foundation for the college mathematics with which they are familiar. It may be useful as supplemental and enrichment material for teaching mathematics in American schools and at home, but for the reasons outlined above, and because the Singapore secondary curriculum does not mesh well with American elementary grade material, is probably not suitable as the source of principal texts.

As mathematicians and applied mathematicians, we feel that a major shortcoming of the two American curricula, ironically, is that they adhered to the 1989 NCTM Curriculum Standards too literally at the expense of the level of the mathematics taught and the mathematical proficiency expected of the students. Responding to criticisms of its earlier Standards, the 2000 NCTM Principles and Standards now raises the bar with respect to arithmetic and computational skills. The Standards now state that sixth through eighth grade students should "develop and analyze algorithms for computing with fractions, decimals and integers, and develop fluency in their use." CMP and MIC do not meet these new standards in the number strand, one of the most fundamental subjects in mathematics. For example, division of fractions is not discussed at all even through 8th grade in CMP and only discussed at the level of "how many 1 1/4's are there in 8 1/2" in MIC. This shortcoming has been a source of criticism from both mathematicians and parents. The developers of the curricula are aware of it, and will probably remedy this deficiency in the future editions. We also expect that in the next edition some of the typographical errors, on which much has been written and which are usually interpreted by critics of these curricula as mathematically incorrect reasoning, will also be corrected. (e.g. Complaints have been raised about "Find the slope and y-intercept for the equation $10 = x - 2.5$", CMP 7th grade *Moving Straight Ahead*, where y was mistakenly printed as 10.)

The impression we gained from comparing the American curricula with the NCTM Principles and Standards is that the curriculum developers take the standards seriously and consequently perhaps adhere to them too literally. Therefore it is incumbent upon the community of mathematicians and mathematics educators to be explicit in providing guidance on the NCTM Principles and Standards. We noted some shortcomings in this respect in the 2000 NCTM Standards which may have implications for the new curriculum revisions from the publishers. To use an example discussed earlier, the 2000 NCTM Standards set a higher standard in the number strand as compared to the 1989 version by requiring "fluency" in computing with fractions. Yet in the guidance following that statement, it appears to suggest that division should be done by repeated subtraction (such as in the cutting ribbon example), which is a flawed algorithm in our opinion and not generalizable to all fractions. Although such a conceptual understanding could be taught as one of the many ways for understanding the meaning of division of fractions, there is the danger that the curriculum developers may interpret this guidance as the definition for "fluency" required and stop at that level. To us, doing division by repeatedly cutting off pieces of a ribbon does not remotely demonstrate "fluency."

The level of the mathematics in both CMP and MIC is not as advanced as that in the Singapore curriculum (with the exception of probability, which is delayed until the 10th grade in Singapore). Some of the mathematics in CMP and MIC has already been covered in Everyday Mathematics, an exemplary elementary school curriculum with which we are familiar. It is also our prediction that students wishing to take calculus before the end of their 12th grade year are likely not to be on track to do so after completing 8th grade CMP or MIC, but would be ready to do so after completing Singapore's *SL2*. We are not advocating that calculus in high school should be a goal for all students, but if this is the desired goal for certain students, the proper supplementation of CMP and MIC at an accelerated pace cannot be ignored. Moreover, we are skeptical about the possibility of maintaining the interest of high-end students while progressing at the pace necessitated by the discovery process, if care is not taken to individualize these discoveries for the students. In schools where student achievement is higher, it may be beneficial to use 8th grade texts for 7th grade students and to use 7th grade texts for the 6th grade students, as is already being done in many schools across the country.

CMP and MIC meet the 2000 NCTM algebra standard, although the mathematical level is much lower than that covered in the Singapore texts. Generalization and abstraction of concepts discovered and learned, which could easily have been included in the curricula, are mostly absent in the American texts. It appears that this may be done deliberately in the authors' attempt to offer easily visualizable problems and concrete examples. For example, CMP's 8th grade algebra text has an excellent introduction to the distributive law of multiplication using the example of calculating the area of a rectangle with two parts. It can be calculated either by first finding the area of each part and then adding them, or by adding the two lengths and two widths and then multiplying the sums to get the total area. By seeing "in context" that the two ways of calculating area are equivalent, a student can then discover the distributive law, without being told that it is true. However, multiplying polynomials that are higher than first order is not covered in the entire curriculum. This could be because it is difficult to come up with a context for multiplying an area by an area, or it could be the result of a decision that the topic is non-essential to a middle school student since it is not explicitly called for by the NCTM Standards. In either case, it is an omission which requires attention for students who wish to be on an accelerated track in high school. Similarly, the division of a polynomial by another polynomial of lower order is not covered, probably because it would have required conceptual understanding of long division at a level not covered by the curricula, which encourages the use of calculators for these problems.

It is not clear to us that a curriculum is the major component of mathematical success for students. The successful implementation of any curriculum depends on a mathematically proficient teacher. As we said in our Introduction, we have avoided the issue of implementation in our comparison studies in Chapters 3 and 4. Instead we have compared the curricula to the 2000 NCTM Principles and Standards under the assumption that ideal conditions exist for their implementation. Many of the criticisms of the new curricula we hear expressed by parents are probably related to poor implementation. It is our judgment that CMP and MIC are more difficult to implement well than a traditional curriculum such as that of Singapore. ("Let us not forget that training *by rote* is a less dangerous weapon in the hands of a teacher of limited mathematical preparation and understanding than are attempts to foster understanding in others that one does not have oneself." Herbert Clemens, 2000, *Notices of the American Mathematical Society*, vol. 74, p. 1074). CMP and MIC also present difficulties to parents who want to help their children, or even to keep track of where they are mathematically. Many familiar landmarks have changed in character or disappeared altogether. A mathematician looking at the materials may initially be frustrated by the fact that the unit titles in CMP, clearly chosen with the intention of making them attractive to middle schoolers, give absolutely no direct information about whether the content is measurement, algebra, geometry, trigonometry, statistics, probability, etc., though the subtitle for each unit is more informative. Similar frustration can result from the fact that in MIC information such as a glossary and index is hidden from the students and parents.

In both MIC and CMP, the lessons are rooted in discovery-based learning. However, it is a truism that teaching in the "Socratic mode" requires more knowledge on the part of the instructor than traditional teaching in the "lecture mode." Since these curricula stress discovery of mathematics by the students, requisite for the success of the curricula is mathematical proficiency on the part of the teachers. In particular, leading students toward mathematical knowledge, without just feeding it to them, requires the teacher to have a broad idea of where the books are leading the students, and of what mathematics precedes and succeeds the current lesson. While each of the teachers' manuals contains an overview that attempts to lay out this very progression, this material is likely only to be useful to teachers with a firm grasp of the underlying mathematics. Teachers will have to judge the mathematical arguments of their students, which will often be novel and unique; they will need to discern the value of algorithms students develop and whether they are sufficiently general to encourage the students to refine them further. This level of mathematical expertise will have to be gained by professional development or with the aid of a mathematical specialist at the school. Alternatively, as some countries have done, we may need to have specialists take over the teaching of mathematics, especially when attempting to implement learning in the "discovery mode." In addition, while a discovery-based format can allow children with different learning styles to engage the material in the manner most useful for them, the teacher must herself be comfortable with the resulting diversity of activity in her classroom and with aiding children who have different needs. Thus we feel that success with these new curricula will require American education schools to educate future teachers with an increased mathematical understanding.

A related comment is that discovery-based learning naturally takes more time than the traditional lecture-then-practice format. If students are to be provided with the chance to experiment, to create, and to test their own hypotheses (some of which will be dead-ends), and to explain their results and their reasoning, more time will have to be allotted to mathematics in the middle grades. This is something which any school district implementing CMP or MIC should take into account.

Finally, we return to the questions raised by the TIMSS study alluded to in our Introduction. Is the Singapore curriculum in mathematics responsible for its students' top ranking in the TIMSS tests? Is the abysmal performance of American middle and high school students in the TIMSS tests attributable to flaws in the American mathematics curricula? Would American students' performances move to the top if only we adopted the Singapore curriculum here? Of course, if the TIMSS tested rote memory, recall of facts, and manipulation skills, Singapore students would have an edge over American students; their curriculum emphasizes practice problems and makes sure that the students attain fluency and computational skills at a level which we judge to be one to two grades higher than their American counterparts. However, the TIMSS tests were designed to test understanding of concepts in addition to competency. The inferences that can be made thus become more murky. On one hand, we must acknowledge that Singapore's educational system – the curriculum, the teachers, the parental support, the social culture, and the strong government support of education – has succeeded in producing students who as a whole understand mathematics at a higher level, and perform with more competence and fluency, than the American students who took the tests. Simply adopting the middle-grades Singapore curriculum is not likely to help American students move to the top, not to mention the fact that the middle-grades Singapore curriculum cannot be adopted without also adopting the elementary Singapore curriculum, and that the teachers at the secondary grades have to receive more training in the subject matter. On the other hand, we doubt very much that any tests, no matter how well designed, can accurately test creativity and independent thinking, qualities which the new American curricula, such as CMP and MIC, strive to foster. Nevertheless, given the fact that the students who took the TIMSS tests probably did not have the benefit of the new curricula, their exceedingly low scores may simply mean that the American students were less well educated in mathematics and science at the time than their Singapore counterparts. The new curricula, and their expected revised versions, may change that. We have found much to like in the new American curricula, especially their emphasis on conceptual understanding and on educating independent thinkers, qualities we value in our society. Can future tests be devised to test gains in these areas? We will watch with great interest the efforts to answer this question.